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On exact solutions of the Schrödinger equation

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Abstract. A new approach has been used to show what kind of potentials we can use to obtain a series of exact solutions in closed forms of the Schrödinger equation for bound states. In particular, the well known Coulombic and oscillator solutions and others are reproduced as special cases.

1. Introduction

The standard quantum mechanical applications of the radial Schrödinger equation were extended to various branches of physics recently (for example, quark physics (cf Quigg and Rosner 1979), laser theory (Haken 1970), field theory in zero dimensions (Kaushal 1974) and nuclear physics (Lai 1983). It is well known that the radial Schrödinger equation must be solved numerically in general, and the complete and non-numerical solution exists in a closed form only for a few forces (Newton 1965). One of the most interesting topics is to search for new methods which can be used to find the exact solutions in closed forms of the Schrödinger equation for more new potentials. Recently, some exact solutions in closed forms have been given for $V(r) = V_0 - 2(N+\frac{1}{2})V_6^{1/2}r + V_6r^6$ (Yang 1979), $V(r) = r^2 + \lambda r^2(1+gr^2)^{-1}$ (Flessas 1981, Lai and Lin 1982, Whitehead *et al* 1982, Znojil 1983), and $V(r) = -r^{-1} \pm 2Ar + 2\lambda^2 r^2$ (Saxena and Varma 1982). (We notice that $V(r) = -r^{-1} \pm 2Ar + 2\lambda^2 r^2$ is just the form shown in (28) of the paper by Yang (1979).)

The aim of the present paper is to improve on the new method introduced by Yang (1979).

2. The method

Let us study the radial Schrödinger equation for an attractive radial potential $V(r)$:

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{l(l+1)}{r^2}R + V(r)R = ER \qquad (1)$$

where the units $2m = \hbar = 1$ are used. If we introduce the function $u(r) = rR(r)$ and a certain new variable $\xi(r)$, we find from (1)

$$u'' + Au' + \xi'^{-2}(E - F)u = 0 \qquad (2)$$

where $u' = du/d\xi$, $A = \xi''\xi'^{-2}$, $\xi' = d\xi/dr$ and

$$F = \frac{l(l+1)}{r^2} + V(r). \tag{3}$$

With an ansatz

$$u = He^{-G} \tag{4}$$

we may write equation (2) in the form

$$H'' - 2(G' - \tfrac{1}{2}A)H' + [(E - F)\xi'^{-2} + G'(G' - A) - G'']H = 0 \tag{5}$$

where primes on both H and G denote differentiation with respect to ξ. It may be emphasised here that both $\xi(r)$ and G also need to be determined. We shall search for all the bound states to (5) which have the elementary forms

$$H(\xi) = \xi^s \sum_{\nu \geq 0} a_\nu \xi^\nu \qquad a_0 \neq 0. \tag{6}$$

2.1. *The simplest mode*

As shown in (5), one simple mode that yields the simplest recursion relation between coefficients of successive terms of the series (6) is seen to be

$$G' - \tfrac{1}{2}A = B_{-2}\xi^{-1} + B_m\xi^{m+1} \tag{7a}$$

$$(E - F)\xi'^{-2} + G'(G' - A) - G'' = -D_{-2}\xi^{-2} + D_m\xi^m \tag{7b}$$

where the constants B_{-2}, B_m, D_{-2}, D_m and m all need to be determined. From (7a) we obtain

$$e^{-G} = \xi'^{-1/2}\xi^{-B_{-2}} \exp\left(-\frac{B_m}{m+2}\xi^{m+2}\right) \tag{8}$$

and

$$u = \xi'^{-1/2}\xi^{s-B_{-2}} \sum_\nu a_\nu \xi^\nu \exp\left(-\frac{B_m}{m+2}\xi^{m+2}\right). \tag{8$'$}$$

As shown in (8$'$), we may let $B_{-2} = 0$ in (7a) since the contribution of B_{-2} may be merged into s, and we shall confine ourselves to the cases with

$$B_m/(m+2) > 0$$
$$\left|\xi'^{-1/2} \exp\left(\frac{B_m}{m+2}\xi^{m+2}\right)\right| \xrightarrow{|\xi| \to \infty} \infty. \tag{8$''$}$$

Substitution of (7) into (5) yields the recursion relation

$$[(\nu + s)(\nu + s - 1) - D_{-2}]a_\nu - \{2[\nu + s - (m+2)]B_m - D_m\}a_{\nu-(m+2)} = 0. \tag{9}$$

It then follows that

$$s(s-1) = D_{-2} \tag{10}$$

or

$$s = \tfrac{1}{2} \pm (\tfrac{1}{4} + D_{-2})^{1/2} \tag{10$'$}$$

Table 1. Some potentials and their respective exact eigensolutions obtained according to mode (7).

Number	$\xi(r)$	Effective potential $F(r)$: (14)	Effective potential $F(r)$: The concrete form	Eigenvalues†	Remarks
1-1	$\xi'\xi^{m+1}=a$, $\xi(r)=[(m+2)a(r+b)]^{1/(m+2)}$	$E+B_m^2a^2$ $-\left(2n+1+\frac{2s-1}{m+2}\right)B_m a(r+b)^{-1}$ $+\left(D_{-2}-\frac{(m+1)(m+3)}{4}\right)$ $\times(m+2)^{-2}(r+b)^{-2}$	F_0 $+F_{-1}(r+b)^{-1}$ $+F_{-2}(r+b)^{-2}$	$E_n=F_0-\frac{1}{4}F_{-1}^2[n+\frac{1}{2}\pm(\frac{1}{4}+F_{-2})^{1/2}]^2$, $n=0,1,2,\ldots$	The well known Coulombic solution will be reproduced with $b=0$, $F_0=0$, $F_{-1}=-Ze^2$, $F_{-2}=l(l+1)$
1-2	$\xi'\xi^{m/2}=a$, $\xi(r)=\left(\frac{m+2}{2}a(r+b)\right)^{2/(m+2)}$	$E-[(2n+1)(m+2)+2s-1]B_m a^2$ $+\frac{1}{4}(m+2)^2B_m^2a^4(r+b)^2$ $+\left[D_{-2}-\frac{m}{4}\left(1+\frac{m}{4}\right)\right]$ $\times 4(m+2)^{-2}(r+b)^{-2}$	F_0 $+F_2(r+b)^2$ $+F_{-2}(r+b)^{-2}$	$E_n=F_0+2[2n+1\pm(\frac{1}{4}+F_{-2})^{1/2}]\sqrt{F_2}$, $n=0,1,2,\ldots$	The well known harmonic oscillator solution will be reproduced with $b=0$, $F_0=0$, $F_2=\frac{1}{2}\mu w^2$, $F_{-2}=l(l+1)$
1-3	$\xi'\xi^{-1}=a$, $\xi(r)=be^{ar}$	$E+(D_{-2}+\frac{1}{4})a^2$ $+B_m^2a^2b^{2(m+2)}e^{2(m+2)ar}$ $-[(2n+1)(m+2)+2s-1]B_m a^2$ $\times b^{m+2}e^{(m+2)ar}$	F_0 $+F_2e^{2gr}$ $+F_1e^{gr}$	$E_n=F_0-g^2(n+\frac{1}{2}+F_1/2g\sqrt{F_2})^2$, $n=0,1,2,\ldots$	The known exact eigen-solutions for Morse potential for $l=0$ will be reproduced with $F_0=0$, $F_1/F_2=-2$, $F_2>0$, $l=0$

† '±' should be selected correctly according to the boundary condition that $R(\xi)$ should be finite at $\xi=0$.

and

$$a_\nu / a_{\nu-(m+2)} \xrightarrow{\nu\to\infty} 2B_m/\nu \tag{11}$$

where '±' in (10′) should be chosen correctly according to the boundary condition that R be finite at $\xi=0$. From (11) we see that if the series (6) does not terminate, then there will be

$$H(\xi) \xrightarrow{|\xi|\to\infty} \xi^\alpha \exp\left(\frac{2B_m}{m+2}\xi^{m+2}\right)$$

$$u(\xi) \xrightarrow{|\xi|\to\infty} \xi'^{-1/2}\xi^\alpha \exp\left(\frac{B_m}{m+2}\xi^{m+2}\right)$$

and therefore the solution cannot be normalised under the condition (8″). As shown in (9), the series $H(\xi)$ will terminate and reduce to a polynomial:

$$H_n(\xi) = \xi^s \sum_{\eta=0}^{n} a_{\eta w}\xi^{\eta w} \tag{12}$$

if

$$D_m = 2(nw+1)B_m \tag{13}$$

where $w=m+2$. It is worth noting that the requirements (12) and (13) will not be necessary for some special cases in which $|\xi|\to\infty$ never happened, but even for those cases the eigensolutions derived hereafter should be correct, though they may be incomplete as shown by number 2-2 and 2-5 in table 2, for example. Substitution of (7*a*) and (13) into (7*b*) give

$$F = E + \xi'^2\{-\tfrac{1}{4}(2A'+A^2) + B_m^2\xi^{2(m+1)} - [(2n+1)w+2s-1]B_m\xi^m + D_{-2}\xi^{-2}\}. \tag{14}$$

This result shows what kind of potentials can be solved exactly with the approach just described.

As an example, if we let

$$\xi'\xi^{m+1} = \text{constant}\times a \tag{15}$$

(which is shown in table 1 as number 1-1) then (15) yields

$$\xi(r) = [(m+2)a(r+b)]^{1/(m+2)} \tag{16}$$

and

$$A = \frac{d}{d\xi}\ln\xi' = -(m+1)\xi^{-1} \tag{16′}$$

where b is an arbitrary constant. Substitution of (15)-(16′) into (14) gives

$$F = E + B_m^2a^2 - \left(2n+1+\frac{2s-1}{m+2}\right)B_m a(r+b)^{-1} + \left(D_{-2} - \frac{(m+1)(m+3)}{4}\right)(m+2)^{-2}(r+b)^{-2} \tag{17}$$

which means that for a known concrete potential of the form

$$F = F_0 + F_{-1}(r+b)^{-1} + F_{-2}(r+b)^{-2} \tag{18}$$

the respective eigensolutions can be obtained easily by the new approach just described above. The comparison between (17) and (18) gives

$$E = F_0 - B_m^2 a^2 \tag{19a}$$

$$B_m a = -F_{-1}\left(2n + 1 + \frac{2s-1}{m+2}\right)^{-1} \tag{19b}$$

$$D_{-2} - \tfrac{1}{4}(m+1)(m+3) = (m+2)^2 F_{-2}. \tag{19c}$$

From (19c) and (10) we obtain

$$s = \tfrac{1}{2} \pm (m+2)(\tfrac{1}{4} + F_{-2})^{1/2}. \tag{20}$$

From the substitution of (20) into (19b) and then (19b) into (19a), we obtain eigenvalues

$$E_n = F_0 - \tfrac{1}{4}F_{-1}^2[n + \tfrac{1}{2} \pm (\tfrac{1}{4} + F_{-2})^{1/2}]^{-2} \qquad n = 0, 1, 2, \ldots . \tag{21}$$

Since the respective $H_n(\xi)$ can be found from (7) trivially, it will be omitted here. As is shown in this example, because the values of $m+2$ and a are not determined uniquely by the comparison, we may choose $m+2=1$ and $a=1$ for convenience. It is apparent to us that the well known Coulombic solution is reproduced if

$$F_0 = 0 \qquad b = 0 \qquad F_{-2} = l(l+1) \qquad F_{-1} = Ze^2.$$

Two other examples belonging to mode (7) are also shown as number 1-2 and 1-3 in table 1. They may be analysed in the same way as number 1-1. Apparently, the well known oscillator solution and the known eigensolutions for $l=0$ for the Morse potential (cf Flugge 1974, p 186) are reproduced as number 1-2 and 1-3, respectively, when suitable parameters are taken. It usually appears that for many potentials with respect to other types of $\xi(r)$ other than those listed in table 1 only one single eigensolution can be provided by this new approach belonging to mode (7).

2.2. *The other simple mode*

Another mode which can yield a simple recursion relation is

$$G' - \tfrac{1}{2}A = (B_{-2}\xi^{-1} + B_m\xi^{m+1})(1 + K_m\xi^{m+2})^{-1} \tag{22a}$$

$$(E-F)\xi'^{-2} + G'(G'-A) - G'' = (-D_{-2}\xi^{-2} + D_m\xi^m)(1 + K_m\xi^{m+2})^{-1} \tag{22b}$$

where the parameter $K_m \neq 0$. From (22a) we obtain

$$e^{-G} = \xi'^{-1/2}\xi^{-B_{-2}}(1 + K_m\xi^{m+2})^{(B_{-2}-b_m)/(m+2)}$$

and therefore

$$u = \xi'^{-1/2}\xi^{s-B_{-2}}(1 + K_m\xi^{m+2})^{(B_{-2}-b_m)/(m+2)} \sum_{\eta \geq 0} a_{\eta w}\xi^{\eta w} \tag{23}$$

where $b_m = B_m/K_m$. We shall confine ourselves to searching for only all the bound states with the elementary form

$$H_n(\xi) = \xi^s \sum_{\eta=0}^{n} a_{\eta w}\xi^{\eta w}. \tag{23'}$$

As shown in (23), we may let $B_{-2}=0$ in (22) since the contribution of B_{-2} can be merged into s and b_m. Substitution of (22) into (5) gives

$$[(\nu+s)(\nu+s-1)-D_{-2}]a_\nu+[(\nu+s-w)(\nu+s-1-w)K_m - 2(\nu+s-w)B_m+D_m]a_{\nu-w}=0. \tag{24}$$

The series (6) reduces to the polynomial (23′) only if

$$D_m=2(nw+s)B_m-(nw+s)(nw+s-1)K_m \tag{25}$$

where s can be found from (4) too, i.e.

$$s=\tfrac{1}{2}\pm(\tfrac{1}{4}+D_{-2})^{1/2}. \tag{25'}$$

Substitution of (22a) into (22b) gives

$$F=E+\left(\frac{\xi'}{\xi}\right)^2\left(-\tfrac{1}{4}(2A'+A^2)\xi^2+\frac{b_m(b_m+w)K_m^2\xi^{2(m+2)}}{(1+K_m\xi^{m+2})^2} + \frac{D_{-2}-[d_m+(w-1)b_m]K_m\xi^{m+2}}{1+K_m\xi^{m+2}}\right) \tag{26}$$

which may be rewritten in the form

$$F=E+\left(\frac{\xi'}{\xi}\right)^2\left(-\tfrac{1}{4}(2A'+A^2)\xi^2+b_m(b_m+1)-d_m+\frac{b_m(b_m+w)}{(1+K_m\xi^{m+2})^2} + \frac{D_{-2}+d_m-b_m(2b_m+w+1)}{1+K_m\xi^{m+2}}\right) \tag{26'}$$

where $d_m=D_m/K_m$. The respective eigensolutions with respect to (26) derived by the new method have been shown in table 2 as number 2-1. Some other examples belonging to mode (22) are listed in table 2 as well.

2.3. *The slightly more complex mode*

Another mode which yields a slightly more complex recursion relation seems to have the form

$$G'-\tfrac{1}{2}A=B_{m_1}\xi^{w-1}+B_{m_2}\xi^{2w-1} \tag{27a}$$

$$(E-F)\xi'^{-2}+G'(G'-A)-G''=-D_{-2}\xi^{-2}+D_{m_1}\xi^{w-2}+D_{m_2}\xi^{2w-2} \tag{27b}$$

or

$$G'-\tfrac{1}{2}A=B_{-2}\xi^{-1}+B_m\xi^{w-1} \tag{28a}$$

$$(E-F)\xi'^{-2}+G'(G'-A)-G'' =(-D_{-2}\xi^{-2}+D_{m_1}\xi^{w-2}+D_{m_2}\xi^{2w-2})(1+K_m\xi^w)^{-1} \tag{28b}$$

etc. As an example, (28) yields

$$[(\nu+s)(\nu+s-1)-2(\nu+s)B_{-2}-D_{-2}]a_\nu +[(\nu+s-w)(\nu+s-1-w)-2(\nu+s-w)(B_m+B_{-2}K_m)+D_{m_1}]a_{\nu-w} +[D_{m_2}-2(\nu+s-2w)B_mK_m]a_{\nu-2w}=0$$

which may be rewritten in detail as follows:

$$[s(s-1)-2sB_{-2}-D_{-2}]a_0=0 \tag{29a}$$

$$[(s+w)(s+w-1)-2(s+w)B_{-2}-D_{-2}]a_w$$
$$+[s(s-1)K_m-2s(B_m+B_{-2}K_m)+D_{m_1}]a_0=0 \tag{29b}$$

$$[(s+2w)(s+2w-1)-2(s+2w)B_{-2}-D_{-2}]a_{2w}$$
$$+[(s+w)(s+w-1)K_m-2(s+w)(B_m+B_{-2}K_m)+D_{m_1}]a_w$$
$$+(D_{m_2}-2sB_mK_m)a_0=0 \tag{29c}$$

...

$$[(s+nw)(s+nw-1)-2(s+nw)B_{-2}-D_{-2}]a_{nw}$$
$$+\{[s+(n-1)w][s+(n-1)w-1]K_m$$
$$-2[s+(n-1)w](B_m+B_{-2}K_m)+D_{m_1}\}a_{(n-1)w}$$
$$+\{D_{m_2}-2[s+(n-2)w]B_mK_m\}a_{(n-2)w}=0 \tag{29d}$$

$$[(s+nw)(s+nw-1)K_m-2(s+nw)(B_m+B_{-2}K_m)+D_{m_1}]a_{nw}$$
$$+\{D_{m_2}-2[s+(n-1)w]B_mK_m\}a_{(n-1)w}=0 \tag{29e}$$

$$[D_{m_2}-2(nw+s)B_mK_m]a_{nw}=0. \tag{29f}$$

(29*a*–*b*) and (29*e*–*f*) are just the requirements for H having the form of a polynomial, and it is shown that $n>0$ is necessary. In order to obtain non-zero solutions, (29*a*) and (29*f*) yield, respectively,

$$s=B_{-2}+\tfrac{1}{2}\pm[(B_{-2}+\tfrac{1}{2})^2+D_{-2}]^{1/2} \tag{29a'}$$

$$D_{m_2}=2(nw+s)B_mK_m. \tag{29f'}$$

Now equations (29*b*–*e*) may be regarded as a set of homogeneous algebraic equations for $a_{\eta w}$. The necessary and sufficient condition that these equations have a solution is that the determinant of their coefficients vanishes. This provides a secular equation which may be written formally in the form

$$f(n, s, D_{m_1}, B_{-2}, B_m, K_m)=0. \tag{30}$$

Substitution of (28*a*) into (28*b*) gives

$$F=E+\left(\frac{\xi'}{\xi}\right)^2\left(-\tfrac{1}{4}(2A'+A^2)\xi^2+B_m^2\xi^{2(m+2)}+[2B_{-2}-(w-1)]B_m\xi^{m+2}\right.$$
$$\left.+B_{-2}(B_{-2}+1)+\frac{D_{-2}-D_{m_1}\xi^{m+2}-D_{m_2}\xi^{2(m+2)}}{1+K_m\xi^{m+2}}\right). \tag{31}$$

Throughout the further analysis according to precedent, it is shown unfortunately that only one single eigensolution can be provided by this new method for the special potentials belonging to this mode only if the known parameters in the concrete expression of the potentials satisfy the condition with respect to (30), except for some special examples such as $V(r)=V_0-2(N+\frac{1}{2})V_6^{1/2}r^2+V_6r^6$ (Yang 1979). We should notice that the known eigensolutions for $V(x)=x^2+\lambda x^2(1+gx^2)^{-1}$ can be provided by this new method as well, according to the mode (28) with $\xi(r)=x$.

Table 2. Some potentials and their respective exact eigensolutions obtained according to mode (22).

Number	$\xi(r)$	Effective potential $F(r)$: (26)	Effective potential $F(r)$: The concrete form
2-1	$\xi'\xi^{m+1}(1+K_m\xi^{m+2})^{-1}=a$, $\xi^{m+2}=K_m^{-1}[be^{(m+2)aK_m r}-1]$	$E+[(b_m+\frac{1}{2})^2-d_m]a^2K_m^2$ $+[-d_m-(m+1)b_m$ $+D_{-2}-\frac{1}{2}(m+1)(m+3)]$ $\times a^2K_m^2[be^{(m+2)aK_m r}-1]^{-1}$ $+[D_{-2}+B_{-2}(B_{-2}+1)+\frac{1}{4}-\frac{1}{4}(m+2)^2]$ $\times a^2K_m^2[be^{(m+2)aK_m r}-1]^{-2}$	F_0 $+F_{-1}(be^{gr}-1)^{-1}$ $+F_{-2}(be^{gr}-1)^{-2}$
2-2	$\xi'\xi^{-1}(1+K_m\xi^{m+2})^{-1}=a$, $\xi^{-(m+2)}=K_m[be^{-(m+2)ar}-1]$	$E+(\frac{1}{4}+D_{-2})a^2+a^2[-\frac{1}{2}(m+2)^2$ $+D_{-2}-d_m-(m+1)b_m]$ $\times[be^{-(m+2)ar}-1]^{-1}$ $+a^2[-\frac{1}{4}(m+2)^2+b_m^2+b_m-d_m]$ $\times[be^{-(m+2)ar}-1]^{-2}$	idem
2-3	$\xi'\xi^{-1}=a$, $\xi=be^{ar}$	$E+[(b_m+\frac{1}{2})^2-d_m]a^2$ $+\dfrac{b_m(b_m+w)a^2}{[1+K_m b^{m+2}e^{(m+2)ar}]^2}$ $+\dfrac{[D_{-2}+d_m-b_m(2b_m+w+1)]a^2}{1+K_m b^{m+2}e^{(m+2)ar}}$	F_0 $+\dfrac{F_{-2}}{(1+\lambda e^{gr})^2}$ $+\dfrac{F_{-1}}{1+\lambda e^{gr}}$
2-4	$\xi'\xi^{m/2}[1+K_m\xi^{m+2}]^{-1}=a$, $\xi^{(m+2)/2}=(-K_m)^{-1/2}$ $\times\tanh(\frac{1}{2}(m+2)a(-K_m)^{1/2}r+b)$	$E+[-\frac{1}{4}(\frac{3}{2}m^2+bm+a)$ $+D_{-2}-d_m-(m+1)b_m]K_ma^2$ $-K_ma^2[-\frac{1}{4}m(1+\frac{1}{4}m)+b_m(b_m+1)-d_m]$ $\times\tanh^2[\frac{1}{2}(m+2)a(-K_m)^{1/2}r+b]$ $-K_ma^2[-\frac{1}{4}m(1+\frac{1}{4}m)+D_{-2}]$ $\times\coth^2[\frac{1}{2}(m+2)a(-K_m)^{1/2}r+b]$	F_0 $+F_2\tanh^2(gr+b)$ $+F_{-2}\coth^2(gr+b)$
2-5	$\xi'\xi^{-1}(1+K_m\xi^{m+2})^{-1/2}=\pm a$, $\sqrt{-K_m}\,\xi^{(m+1)/2}=$ $=\mathrm{sech}(\frac{1}{2}(m+2)ar+b)$	$E+a^2[D_{-2}+d_m-b_m(2b_m+w+1)+\frac{1}{2}w]$ $+a^2[(b_m+\frac{1}{2})^2+\frac{1}{8}w-d_m]$ $\times\tanh^2(\frac{1}{2}(m+2)ar+b)$ $+a^2[(b_m+\frac{1}{2}w)^2-\frac{1}{16}w^2]$ $\times\coth^2(\frac{1}{2}(m+2)ar+b)$	idem
2-6	$\xi'\xi^{m/2}(1+K_m\xi^{m+2})^{-1/2}=a$, $\xi^{(m+2)/2}=K_m^{-1/2}$ $\times\sinh[\frac{1}{2}(m+2)\sqrt{K_m}\,ar+b]$	$E+K_ma^2\{[(b_m+\frac{1}{2})^2-d_m]$ $\times\coth^2[\frac{1}{2}w\sqrt{K_m}\,ar+b]$ $+[D_{-2}+d_m-b_m(b_m+1)-\frac{1}{8}w^2]$ $\times\mathrm{cosech}^2[\frac{1}{2}w\sqrt{K_m}\,ar+b]$ $-[b_m(b_m+w)+3w^2/1b]$ $\times\mathrm{sech}^2[\frac{1}{2}w\sqrt{K_m}\,ar+b]\}$	F_0 $+F_1\,\mathrm{cosech}^2(gr+b)$ $-F_2\,\mathrm{sech}(gr+b)$

† '±' should be chosen correctly according to the boundary condition that R should be finite at $\xi=0$.
An upper limit usually exists for n according to the boundary condition that R should be finite as $\xi\to\infty$.

Eigenvalues†	Remarks
$E_n = F_0 + F_{-1} - F_{-2} - \left[\frac{F_{-2} - F_{-1}}{2g(n+s)} + \frac{1}{2}g(n+s)\right]^2$, $s = \frac{1}{2} \pm (\frac{1}{4} + g^{-2}F_{-2})^{1/2}$	(1) Hulthén potential will emerge if $l=0$, $b=1$, $F_{-2}=0$ and $F_0 = F_{-1} = V_0$ (cf Flügge 1974, problem 68). (2) Wood-Saxon potential will energe if $l=0$, $b=e^{-R/a}$, $g=1/a$, $F_{-1}=-1$ and $F_0=F_{-2}=0$ (cf Flügge 1974, problem 64).
$E_n = F_0 - \frac{1}{4}g^2 - F_{-1} + F_{-2} - g[n+s\pm(g^{-2}F_{-2}+\frac{1}{2})^{1/2}]^2$, $s = \frac{g^{-2}(F_{-2}+F_{-1}) - [n \pm (g^{-2}F_{-2}+\frac{1}{2})^{1/2}]^2}{2[n\pm(g^{-2}F_{-2}+\frac{1}{2})^{1/2}]+1}$	(3) The eigensolutions for a potential with $F_0=0$, $F_{-1}=-(\lambda-\mu)$ and $F_{-2}=\mu$ had been analysed by Myhrman (1980). (4) The eigensolutions listed in number 2-1 and 2-2 should be complementary to one another.
$E_n = F_0 + F_{-2} - F_{-1} - \frac{1}{4}g^2 - s(s-1)g^2$, $s = \frac{1}{-(n+\frac{1}{2}) \pm (\frac{1}{4}g^{-2}F_{-2})^{1/2}} \times \left(\frac{2F_{-2}-F_{-1}}{g^2} + \lambda^2 \pm 2n(\frac{1}{4}+g^{-2}F_{-2})^{1/2}\right)$	
$E_n = F_0 + F_2 + F_{-2} - \left\{\begin{array}{l} g^2(2n+1)^2 \\ g^2[2n+1\pm 2(\frac{1}{4}+g^{-2}F_{-2})^{1/2}]^2 \end{array}\right.$	(1) Pöschl-Teller potential and modified Pöschl-Teller potential will emerge when suitable parameters are taken (cf Flügge 1974, problems 38 and 39). (2) The eigensolutions listed in number 2-4 and 2-5 should be complementary to one another.
$E_n = F_0 + F_2 + F_{-2} - \frac{13}{4}g^2 - s(s-1)g^2$, $s = \frac{1}{2} - 2n \pm (g^{-2}F_2 - \frac{1}{4})^{1/2}$	
$E_n = F_0 - g^2[2n+1\pm(\frac{1}{4}+g^{-2}F_2)^{1/2} \pm (\frac{1}{2}+g^{-2}F_1)^{1/2}]^2$	(1) Can also be extended to following concrete potentials: (i) $F_0 + F_3\coth^2(gr+b) + F_4\,\mathrm{sech}^2(gr+b)$ (ii) $F_0 + F_5\coth^2(gr+b) + F_6\tanh^2(gr+b)$ (iii) $F_0 + F_7\,\mathrm{cosech}^2(gr+b) + F_8\tanh^2(gr+b)$

3. Summary

A new method has been developed for eigenvalue problems in quantum mechanics. By using this method, not only are almost all the known exact solutions of the Schrödinger equation for the respective potentials reproduced, but also more new exact solutions have been found. It is apparent that there are many potentials for which only one single eigensolution can be provided by this new method. The new method can be applied to the Dirac equation too, and some new results will be reported in a further paper.

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